Math 150, ***Lecture Notes- Bonds*** **Name____________**

Section 5.7 *Inverse Trigonometric Functions: Integration*

Integrals Involving Inverse Trigonometric Functions

The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}.$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair. It is conventional to use $\arcsin x$ as the antiderivative of $1/\sqrt{1-x^2}$, rather than $-\arccos x$. The next theorem gives one antiderivative formula for each of the three pairs.

THEOREM 5.17 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Ex.1 Integration with Inverse Trigonometric Functions

a. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{\sqrt{2^2-x^2}}$

$= \arcsin\left(\frac{x}{2}\right) + C$

$u^2 = x^2$
$u = x$
$du = dx$

$a^2 = 4$
$a = 2$

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THEOREM 5.17 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$ **2.** $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

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Ex.1 Integration with Inverse Trigonometric Functions

a. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{1}{\sqrt{2^2-x^2}}\,dx$

$= \arcsin\left(\frac{x}{2}\right) + C$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

b. $\displaystyle\int \frac{dx}{2+9x^2} = \int \frac{dx}{(\sqrt{2})^2 + (3x)^2}$

$u = 3x$, $\frac{du}{dx} = 3$

$a = \sqrt{2}$

$du = \frac{du}{dx} \cdot dx$

$du = 3 \cdot dx$

$\boxed{\frac{du}{3} = dx}$

$$= \int \frac{1}{(\sqrt{2})^2 + u^2}\left(\frac{du}{3}\right)$$

$$= \frac{1}{3} \cdot \int \frac{1}{(\sqrt{2})^2 + u^2}\,du$$

$$= \frac{1}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\right] + C$$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{3x}{\sqrt{2}}\right) + C$$

$a^2 = \frac{2}{9}$

$a = \frac{\sqrt{2}}{3}$

$$\int \frac{1}{2+9x^2}\,dx$$

$$\int \left(\frac{1}{2+9x^2}\right)\left(\frac{\frac{1}{9}}{\frac{1}{9}}\right)dx = \frac{1}{9}\int \frac{1}{\frac{2}{9}+x^2}\,dx = \frac{1}{9}\cdot\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x}{\frac{\sqrt{2}}{3}}\right)\right] + C$$

$$= \frac{1}{9}\cdot\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x}{\sqrt{2}}\right) + C$$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{3x}{\sqrt{2}}\right) + C$$

c. $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 9}} =$

let

$u^2 = 4x^2$

$u = 2x$

$\frac{du}{dx} = 2$

$du = \frac{du}{dx} \cdot dx$

$du = 2 \cdot dx$

$\frac{du}{2} = dx$

$a^2 = 9$

$a = 3$

$u = 2x$

$\frac{u}{2} = x$

$$= \int \frac{1}{\frac{u}{2}\sqrt{u^2 - 3^2}}\left(\frac{du}{2}\right)$$

$$= \int \frac{1}{u\sqrt{u^2 - 3^2}}\,du$$

$$= \frac{1}{3}\sec^{-1}\left(\frac{|u|}{3}\right) + C$$

$$= \frac{1}{3}\sec^{-1}\left(\frac{|2x|}{3}\right) + C$$

Ex.2 Integration by Substitution

Find $\int \frac{dx}{\sqrt{e^{2x}-1}}$.

$= \int \frac{1}{\sqrt{(e^x)^2-1}}\,dx$

$= \int \frac{1}{\sqrt{u^2-1}}\left(\frac{du}{u}\right)$

$= \int \frac{1}{u\sqrt{u^2-1}}\,du$

$= \frac{1}{1}\sec^{-1}\left(\frac{|u|}{1}\right) + C$

$= \sec^{-1}|e^x| + C$

$= \sec^{-1}(e^x) + C$

Let $u = e^x$

$u^2 = (e^x)^2 = e^{2x}$

$\frac{du}{dx} = e^x$

$du = \frac{du}{dx}\cdot dx$

$du = e^x\cdot dx$

$du = u\cdot dx$

$\frac{du}{u} = dx$

$a^2 = 1$

$a = 1$

Ex.3 Rewriting as the Sum of Two Quotients

Find $\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

$\int \frac{x+2}{\sqrt{4-x^2}}dx = \int \frac{x}{\sqrt{4-x^2}}dx + \int \frac{2}{\sqrt{4-x^2}}dx$

$= \int \frac{1}{\sqrt{u}}\cdot\left(\frac{-du}{2}\right) + 2\int \frac{1}{\sqrt{2^2-x^2}}dx$

$= -\frac{1}{2}\int u^{-1/2}du + 2\cdot\left[\arcsin\left(\frac{x}{2}\right)\right]$

$= -\frac{1}{2}\left[\frac{2}{1}\cdot u^{1/2}\right] + 2\arcsin\left(\frac{x}{2}\right) + C$

$= -u^{1/2} + 2\arcsin\left(\frac{x}{2}\right) + C$

$= -\sqrt{4-x^2} + 2\arcsin\left(\frac{x}{2}\right) + C$

Let $u = 4-x^2$

$\frac{du}{dx} = -2x$

$du = \frac{du}{dx}\cdot dx$

$du = -2x\cdot dx$

$\frac{-du}{2} = x\cdot dx$

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \end{aligned}$$

Ex.4 Completing the Square

Find $\displaystyle\int \frac{dx}{x^2 - 4x + 7}$.

$$= \int \frac{1}{(x-2)^2 + (\sqrt{3})^2}\,dx$$

$$= \int \frac{1}{u^2 + (\sqrt{3})^2}\,du$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right) + C$$

$$x^2 - 4x + 7$$

$$= (x^2 - 4x) + 7$$

$$\left[\frac{1}{2}(-4)\right]^2 = (-2)^2 = 4$$

$$= (x^2 - 4x + 4) + 7 - 4$$

$$= (x-2)(x-2) + 3$$

$$= (x-2)^2 + 3$$

$$= (x-2)^2 + (\sqrt{3})^2$$

Let $u = x - 2$

$$\frac{du}{dx} = 1$$

$$du = \cancel{\frac{du}{dx}} \cdot dx$$

$$du = 1 \cdot dx$$

$$du = dx$$

$$a^2 = (\sqrt{3})^2$$

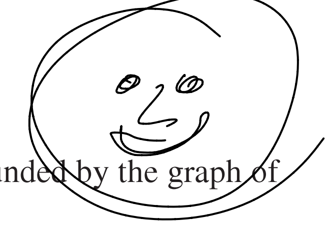
$$a = \sqrt{3}$$

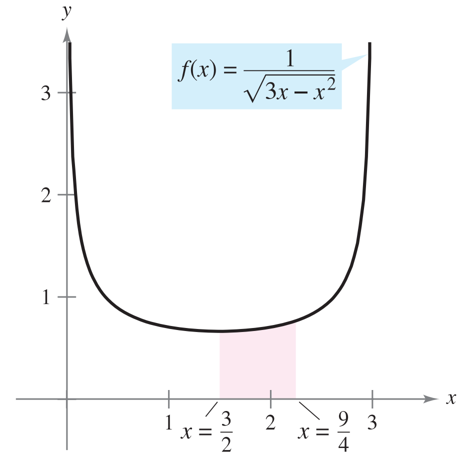
Ex.5 Completing the Square

Find the area of the region bounded by the graph of

$$f(x) = \frac{1}{\sqrt{3x - x^2}}$$

the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.





The area of the region bounded by the graph of f, the x-axis, $x = \frac{3}{2}$, and $x = \frac{9}{4}$ is $\pi/6$.

Figure 5.34

$$\text{Area} = \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx$$

$$= \int_{3/2}^{9/4} \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - 3/2\right)^2}}\, dx$$

$$= \int_{u=0}^{u=3/4} \frac{1}{\sqrt{a^2 - u^2}}\, du$$

$$= \left[\sin^{-1}\left(\frac{u}{a}\right)\right]_0^{3/4}$$

$$= \sin^{-1}\left(\frac{3/4}{3/2}\right) - \sin^{-1}\left(\frac{0}{3/2}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

$u = x - \frac{3}{2}$

$\frac{du}{dx} = 1$

$du = \frac{du}{dx} \cdot dx$

$du = 1 \cdot dx$

$du = dx$

If $x = 3/2$
$u = \frac{3}{2} - \frac{3}{2}$
$u = 0$

If $x = \frac{9}{4}$
$u = \frac{9}{4} - \frac{3}{2}$
$u = \frac{9}{4} - \frac{6}{4}$
$u = \frac{3}{4}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$
$\sin\theta = \frac{1}{2}$
$\theta = \pi/6$

$$3x - x^2 = -x^2 + 3x$$
$$= -(1x^2 - 3x)$$
$$= -\left(x^2 - 3x + \frac{9}{4}\right) + \frac{9}{4}$$
$$\left(\frac{-3}{2}\right)^2 = \frac{9}{4}$$
$$= -\left(x - \frac{3}{2}\right)^2 + \frac{9}{4}$$
$$= \frac{9}{4} - \left(x - \frac{3}{2}\right)^2$$
$$= \left(\frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2$$

$$\frac{3}{4} \div \frac{3}{2} = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

Review of Basic Integration Rules

You have now completed the introduction of the **basic integration rules.** To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory.

Basic Integration Rules ($a > 0$)

1. $\displaystyle\int kf(u)\,du = k\int f(u)\,du$
2. $\displaystyle\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\displaystyle\int du = u + C$
4. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\displaystyle\int \frac{du}{u} = \ln|u| + C$
6. $\displaystyle\int e^u\,du = e^u + C$
7. $\displaystyle\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$
8. $\displaystyle\int \sin u\,du = -\cos u + C$
9. $\displaystyle\int \cos u\,du = \sin u + C$
10. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$
11. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$
12. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\displaystyle\int \sec^2 u\,du = \tan u + C$
15. $\displaystyle\int \csc^2 u\,du = -\cot u + C$
16. $\displaystyle\int \sec u \tan u\,du = \sec u + C$
17. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$
18. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
19. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
20. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

You can learn a lot about the nature of integration by comparing this list with the summary of differentiation rules given in the preceding section. For differentiation, you now have rules that allow you to differentiate *any* elementary function. For integration, this is far from true.

The integration rules listed above are primarily those that were happened on during the development of differentiation rules. So far, you have not learned any rules or techniques for finding the antiderivative of a general product or quotient, the natural logarithmic function, or the inverse trigonometric functions. More importantly, you cannot apply any of the rules in this list unless you can create the proper du corresponding to the u in the formula. The point is that you need to work more on integration techniques, which you will do in Chapter 8. The next two examples should give you a better feeling for the integration problems that you *can* and *cannot* do with the techniques and rules you now know.

Ex.6 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}}$ **b.** $\displaystyle\int \frac{x\,dx}{\sqrt{x^2 - 1}}$ **c.** $\displaystyle\int \frac{dx}{\sqrt{x^2 - 1}}$

$\sec^{-1}(x)$ $u = x^2 - 1$

Ex.7 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x \ln x}$ **b.** $\displaystyle\int \frac{\ln x \, dx}{x}$ **c.** $\displaystyle\int \ln x \, dx$

$u = \ln(x)$ $u = \ln x$?

Ex.8 Evaluate the Integral

Find $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

Ex.9 Evaluate the Integral

Find $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\, dx$